

Conditioning diffusions with respect to partial observations

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Abstract

In this paper, we prove a result of equivalence in law between a diffusion conditioned with respect to partial observations and an auxiliary process. By partial observations we mean coordinates (or linear transformation) of the process at a finite collection of deterministic times. Apart from the theoretical interest, this result allows to simulate the conditioned diffusion through Monte Carlo's method, using the fact that the auxiliary process is easy to simulate.

Keywords: Conditioned diffusion, Partial observations, Simulation

1 Introduction

We are interested in multidimensional diffusions solutions of stochastic differential equations (SDE's) generated by a Brownian motion. For a n -dimensional diffusion solution on $[0, T]$ of the following

$$dx_t = b_t(x_t)dt + \sigma_t(x_t)dw_t, \quad x_0 = u \quad (1)$$

where w is a n -dimensional Brownian motion, it is known (see e.g. [7]) that its conditional law $\mathcal{L}(x|x_T = v)$ is given by the law of a bridge process (as extension of Brownian bridge) y solution of

$$dy_t = b_t(y_t)dt + \sigma_t(y_t)d\tilde{w}_t + \sigma_t(y_t)\sigma_t(y_t)^*\nabla_z \log p_{t,T}(z, v)\big|_{z=y_t} dt, \quad y_0 = u$$

where \tilde{w} is a Brownian motion and $p_{s,t}(z, \cdot)$ is the density of x_t knowing $x_s = z$. But in most cases this density is not explicitly known so that we are not able to simulate it easily. For practical purposes, *e.g.* parameter estimation of diffusion processes, simulation of paths corresponding to the conditional law is needed.

In their paper [4], B.Delyon and Y.Hu studied the following equation on $[0, T]$

$$dy_t = b_t(y_t)dt - \frac{y_t - v}{T - t}dt + \sigma_t(y_t)d\tilde{w}_t, \quad y_0 = u. \quad (2)$$

where \tilde{w} is a n -dimensional Brownian motions. Under adequate assumptions, the process y is unique on $[0, T]$, $\lim_{t \rightarrow T} y_t = v$, a.s. and for all positive function f in $C([0, T], \mathbb{R}^n)$ we have

$$\mathbb{E}[f(x)|x_T = v] = C\mathbb{E}[f(y)R(y)]$$

where R is a functional of whole path y on $[0, T]$. The quantity $R(y)$ is computable knowing parameters b, σ, T and v . The constant C is unknown, but in practice the conditional law is estimated through

$$\mathbb{E}[f(x)|x_T = v] \simeq \frac{\sum_i f(y^i)R(y^i)}{\sum_i R(y^i)}$$

where each y^i is an independant sample of (2). In this case, we call the process y a bridge even if y does not have the right targeted law. If $b = 0$ and $\sigma = I_n$ (identity n -dimensional matrix), the process x is a n -dimensional Brownian motion and process y is a n -dimensional Brownian bridge so that $C = R = 1$. This theorem applies in the case of more than one observation. The Markov property indeed implies that the conditional law is the tensor product of each bridge.

The aim of this paper is to extend this result to solve this problem with only partial observations. The previous remark does not apply; indeed we have to treat simultaneously all conditionings. To give an idea, let $w = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$ be a 2-dimensional Brownian motion. The law of w conditioned on $w_S^1 = u$ and $w_T^2 = v$ with $S < T$ is given by that one of y solution of

$$dy_t = d\tilde{w}_t - \left(\frac{y_t^1 - u}{S - t} \mathbf{1}_{t < S} + \frac{y_t^2 - v}{T - t} \mathbf{1}_{t < T} \right) dt, \quad y_0 = u$$

each coordinate is a Brownian bridge.

Let us define our observations. At each deterministic positive observation time of the sequence $0 < T_1 < \dots < T_k < \dots < T_N = T$, we get a partial information given by a linear transformation of x_{T_k} , $L_k x_{T_k}$, where L_k is a deterministic matrix in $M_{m_k, n}(\mathbb{R})$ whose m_k rows form an orthonormal family. So that our aim is to be able to describe the conditional law $\mathcal{L}(x | (L_k x_{T_k} = v_k)_{1 \leq k \leq N})$ where v_k is an arbitrary deterministic m_k -dimensional vector.

We define process y to be the solution of

$$\begin{cases} dy_t = b_t(y_t)dt + \sigma_t(y_t)d\tilde{w}_t - \sum_{k=1}^N P_t^k(y_t) \frac{y_t - u_k}{T_k - t} \mathbf{1}_{(T_k - \varepsilon_k, T_k)}(t)dt \\ y_0 = u \end{cases} \quad (3)$$

where for all time t , all vector z and for all $1 \leq k \leq N$, the matrix $P_t^k(z)$ is an oblique projection and u_k is any vector satisfying $L_k u_k = v_k$. The correction term operates only on the interval $(T_k - \varepsilon_k, T_k)$ where $T_k - \varepsilon_k < T_k$ for technical reasons. We will show that with a good choice for those projections (see Equation (6)) we have the following equivalence in law

$$\mathcal{L}(x | (L_k x_{T_k} = v_k)_{1 \leq k \leq N}) \sim \mathcal{L}(y)$$

with an explicit density (Theorem 1 below).

In this paper a first part is devoted to the study of general bridges which will provide us the good candidate whose law is absolutely continuous with respect to targeted one. The second one provides the main result. Some properties and proofs are postponed in the appendix to ease the reading.

Notations For the sake of readability, we choose not to specify arguments when not necessary. For example (1) becomes

$$dx_t = b_t dt + \sigma_t dw_t$$

For all z , the matrix $a_t(z)$ is defined by

$$a_t(z) = \sigma_t(z) \sigma_t(z)^*$$

we suppose that there exists a positive number ρ such that for all (t, z)

$$\rho^{-1} I_n < a_t(z) < \rho I_n$$

in the sense of symmetric matrices, where I_n is the n -dimensional identity matrix. The function a^{-1} is defined by

$$\begin{aligned} a^{-1} : [0, T] \times \mathbb{R}^n &\rightarrow M_n(\mathbb{R}^n) \\ (t, x) &\mapsto (a_t(x))^{-1} \end{aligned}$$

We define the infimum of all the ε_k

$$\varepsilon_0 = \min_k \{\varepsilon_k\}$$

2 Bridges and bridges approximations

Bridges

We recall that a bridge is defined as a solution of (3)

$$\begin{cases} dy_t = b_t(y_t)dt + \sigma_t(y_t)d\tilde{w}_t - \sum_{k=1}^N P_t^k(y_t) \frac{y_t - u_k}{T_k - t} \mathbf{1}_{(T_k - \varepsilon_k, T_k)}(t)dt \\ y_0 = u_0 \end{cases}$$

We assume that the deterministic parameters b and σ are $C_b^{1,2}$ functions (bounded with bounded derivatives). We assume that

$$(t, z) \mapsto P_t^k(z)$$

is a $C_b^{1,2}$ function and that for any z

$$L_k P_t^k(z) = L_k \quad \text{and} \quad \ker(L_k) = \ker(P_t^k(z)) \quad (4)$$

First of all, a lemma to describe the behaviour of process y

Lemma 1. *The SDE (3) admits a unique solution on $[0, T]$ in the absolute convergence's sense meaning that*

$$\int_{T_k - \varepsilon_k}^{T_k} \frac{\|P_t^k(y_t)(y_t - u_k)\|}{T_k - t} dt < +\infty$$

For all k we have $L_k y_{T_k} = L_k u_k$ almost surely (a.s.). Moreover for $T_k - \varepsilon_k < t < T_k$, $\|L_k(y_t - u_k)\| \leq C_k(\omega)(T_k - t) \log \log[(T_k - t)^{-1} + e]$ a.s., where C_k is a positive random variable.

Proof. Let us remark that for times in $[T_{k-1}, T_k]$ (with $T_0 = 0$) the SDE (3) becomes

$$dy_t = b_t dt - P_t^k \frac{y_t - u_k}{T_k - t} \mathbf{1}_{(T_k - \varepsilon_k, T_k)}(t) dt + \sigma(y_t) d\tilde{w}_t$$

So that we may reduce the proof to the study of (3) with only one observation time, but we here have to consider random initial conditions. If unicity holds it will lead to the result by concatenation. The proof in the case $N = 1$ is given in the appendix with Lemma 6. \square

Bridges approximations

We now introduce approximations that will be useful in the proof of the main result in next section. Let $0 < \varepsilon < \varepsilon_0$, we set

$$dy_t^\varepsilon = b_t^\varepsilon(y_t^\varepsilon)dt + \sigma_t(y_t^\varepsilon)d\tilde{w}_t - \sum_k P_t^k(y_t^\varepsilon) \frac{y_t^\varepsilon - u_k}{T_k - t} \mathbf{1}_{(T_k - \varepsilon_k, T_k - \varepsilon)}(t)dt, \quad y_0^\varepsilon = u_0 \quad (5)$$

The only difference with the Bridge Equation (3) is that each correction term is stopped from a distance ε from the observation time.

Lemma 2. *There exists a constant $0 < \kappa < 1$ such that*

$$\sup_{t \in [0, T]} \mathbb{E}[\|y_t^\varepsilon - y_t\|^2] \leq C\varepsilon^\kappa$$

for all $0 < \varepsilon < (\varepsilon_0 \wedge 1)$ where C is a positive constant. The numbers C and κ depend on T , N , the $(\varepsilon_k)_k$, the $(A_k)_k$, and the bounds for b and σ .

Proof. Given in the appendix, the proof uses classical techniques and auxiliary processes each defined on $[T_{k-1}, T_k]$. \square

3 Result in the case of partial observation

Case where b is bounded

We aim to obtain a Delyon&Hu-type theorem that gives absolute continuity of process x solution of (1) conditioned on observations $(L_k x_{T_k} = v_k)_{1 \leq k \leq N}$ with respect to a bridge process y solution of (3).

We now consider a peculiar projection P , for all k and z

$$P_t^k(z) = a_t(z) L_k^* (L_k a_t(z) L_k^*)^{-1} L_k \quad (6)$$

We set

$$A_t^k(z) = (L_k a_t(z) L_k^*)^{-1}$$

and also

$$\beta_t(z)^k = \sigma_t(z)^* L_k^* A_t^k(z) \quad \text{and} \quad \eta_k(z) = \sqrt{\det(A_t^k(z))}$$

Let us remark that

$$\beta_t^k(z)^* \beta_t^k(z) = A_t^k(z) \quad \text{and} \quad L_k \sigma_t(z) \beta_t^k(z) = I_{m_k} \quad (7)$$

where I_{m_k} is the identity m_k -dimensional matrix. Here are both systems we now consider

$$\begin{aligned} dx_t &= b_t(x_t) dt + \sigma_t(x_t) dw_t, \quad x_0 = u \\ dy_t &= b_t(y_t) dt + \sigma_t(y_t) d\tilde{w}_t - \sum_{k=1}^N \sigma_t(y_t) \beta_t^k(y_t) \frac{L_k y_t - v_k}{T_k - t} \mathbf{1}_{(T_k - \varepsilon_k, T_k)} dt, \quad y_0 = u \end{aligned} \quad (8)$$

The result is the following

Theorem 1. Suppose b , σ and a^{-1} to be $C_b^{1,2}$ -functions. Then for any bounded continuous function f

$$\begin{aligned} &\mathbb{E}[f(x) | (L_k x_{T_k} = v_k)_{1 \leq k \leq N}] \\ &= C \mathbb{E} \left[f(y) \prod_{k=1}^N \eta_k(y_{T_k}) \exp \left\{ - \frac{\|\beta_{T_k - \varepsilon_k}^k(L_k y_{T_k - \varepsilon_k} - v_k)\|^2}{2\varepsilon_k} + \int_{T_k - \varepsilon_k}^{T_k} - \frac{(L_k y_s - v_k)^* L_k b_s(y_s) ds}{T_k - s} \right. \right. \\ &\quad \left. \left. - \frac{(L_k y_s - v_k)^* d(A_t^k(y_t)) (L_k y_s - v_k)}{2(T_k - s)} - \sum_{1 \leq i, j \leq m_k} \frac{d\langle A_{i,j}^k(y), (L_k y - v_k)_i (L_k y - v_k)_j \rangle_s}{2(T_k - s)} \right\} \right] \quad (9) \end{aligned}$$

where C is a positive constant.

Proof. This one consists in using approximations y^ε solutions of (5) of process y solution of (8). Thanks to Girsanov's theorem, we are able to obtain an equality for all bounded continuous function f

$$\mathbb{E}[f(x) G^\varepsilon(x)] = \mathbb{E}[f(y^\varepsilon) H^\varepsilon(y^\varepsilon)]$$

where $G^\varepsilon/H^\varepsilon$ is the density given by Girsanov's theorem. We want to prove that with a good choice for G^ε and H^ε , the lefthand member of the last inequality converges to the conditional expectation, and the righthand one converges to what appears in the Theorem 1.

We set for all $z \in \mathbb{R}^n$

$$h_t^\varepsilon(z) = \sum_{k=1}^N \beta_t^k(z) \frac{v_k - L_k z}{T_k - t} \mathbf{1}_{(T_k - \varepsilon_k, T_k - \varepsilon)}(t)$$

Then for all bounded continuous function f

$$\mathbb{E}[f(y^\varepsilon)] = \mathbb{E}[f(x) \exp \{ - \int_0^T h_t^\varepsilon(x_t)^* dw_t + \frac{1}{2} \|h_t^\varepsilon(x_t)\|^2 dt \}]$$

We are looking for a different expression of the argument of the exponential function. We use Itô's formula for $T_k - \varepsilon_k < t < T_k - \varepsilon$ and use (7) to get

$$\begin{aligned} d \left(\frac{\|\beta_t^k(x_t)(L_k x_t - v_k)\|^2}{T_k - t} \right) &= \frac{2(L_k x_t - v_k)^* A_t^k(x_t) L_k dx_t}{T_k - t} + \frac{\|\beta_t^k(x_t)(L_k x_t - v_k)\|^2}{(T_k - t)^2} dt + \frac{m_k}{T_k - t} dt \\ &\quad + \frac{(L_k x_t - v_k)^* d(A_t^k(x_t)) (L_k x_t - v_k)}{T_k - t} + \sum_{1 \leq i, j \leq m_k} \frac{d\langle A_{i,j}^k(x), (L_k x - v_k)_i (L_k x - v_k)_j \rangle_t}{T_k - t} \end{aligned}$$

The k^{th} term of $(h_t^\varepsilon)^* dw_t$ coming from that one in dx_t is now isolated

$$\begin{aligned} & -\frac{2(L_k x_t - v_k)^* A_t^k \sigma_t dw_t}{T_k - t} - \frac{\|\beta_t^k(L_k x_t - v_k)\|^2}{(T_k - t)^2} dt = -d\left(\frac{\|\beta_t^k(L_k x_t - v_k)\|^2}{T_k - t}\right) + \frac{m_k}{T_k - t} dt \\ & + \frac{2(L_k x_t - v_k)^* A_t^k b_t dt}{T_k - t} + \frac{(L_k x_t - v_k)^* dA_t^k(L_k x_t - v_k)}{T_k - t} + \sum_{1 \leq i, j \leq m_k} \frac{d\langle A_{i,j}^k, (L_k x - v_k)_i (L_k x - v_k)_j \rangle_t}{T_k - t} \end{aligned} \quad (10)$$

Since we have

$$\|h_t^\varepsilon\|^2 dt = \sum_k \frac{\|\beta_t^k(L_k x_t - v_k)\|^2}{(T_k - t)^2} \mathbf{1}_{(T_k - \varepsilon_k, T_k - \varepsilon)}(t) dt$$

and

$$(h_t^\varepsilon)^* dw_t = - \sum_k \mathbf{1}_{(T_k - \varepsilon_k, T_k - \varepsilon)}(t) \frac{(L_k x_t - v_k)^* A_t^k \sigma_t dw_t}{T_k - t}$$

we obtain $-2(h_t^\varepsilon)^* dw_t - \|h_t^\varepsilon\|^2 dt$ adding the terms given by (10). Finally, it leads us to a new expression for the density given by Girsanov's theorem

$$\begin{aligned} \mathbb{E}[f(y^\varepsilon)] &= \mathbb{E}\left[\exp\left\{\sum_{k=1}^N -\frac{\|\beta_{T_k - \varepsilon}^k(L_k x_{T_k - \varepsilon} - v_k)\|^2}{2\varepsilon} + \frac{\|\beta_{T_k - \varepsilon_k}^k(L_k x_{T_k - \varepsilon_k} - v_k)\|^2}{2\varepsilon_k}\right.\right. \\ & \quad \left. + \int_{T_k - \varepsilon_k}^{T_k - \varepsilon} \frac{(L_k x_t - v_k)^* A_t^k b_t dt}{T_k - t} + \frac{m_k}{2(T_k - t)} dt \right. \\ & \quad \left. + \frac{(L_k x_t - v_k)^* dA_t^k(L_k x_t - v_k)}{2(T_k - t)} + \sum_{1 \leq i, j \leq m_k} \frac{d\langle A_{i,j}^k, (L_k x - v_k)_i (L_k x - v_k)_j \rangle_t}{2(T_k - t)}\right\}\Big] \end{aligned}$$

In an equivalent way, even if it means changing f

$$\mathbb{E}[f(y^\varepsilon)\varphi^\varepsilon] = \mathbb{E}[f(x)\psi^\varepsilon] \quad (11)$$

with

$$\begin{aligned} \varphi^\varepsilon := \varphi^\varepsilon(y^\varepsilon) &= \prod_{k=1}^N \varepsilon_k^{-\frac{m_k}{2}} \eta_k^\varepsilon(y_{T_k - \varepsilon}^\varepsilon) \exp\left\{\sum_{k=1}^N -\frac{\|\beta_{T_k - \varepsilon_k}^k(L_k y_{T_k - \varepsilon_k}^\varepsilon - v_k)\|^2}{2\varepsilon_k} + \int_{T_k - \varepsilon_k}^{T_k - \varepsilon} -\frac{(L_k y_t^\varepsilon - v_k)^* A_t^k b_t dt}{T_k - t} \right. \\ & \quad \left. - \frac{(L_k y_t^\varepsilon - v_k)^* dA_t^k(L_k y_t^\varepsilon - v_k)}{2(T_k - t)} - \sum_{1 \leq i, j \leq m_k} \frac{d\langle A_{i,j}^k, (L_k y^\varepsilon - v_k)_i (L_k y^\varepsilon - v_k)_j \rangle_t}{2(T_k - t)}\right\} \end{aligned} \quad (12)$$

and

$$\psi^\varepsilon = C^\varepsilon \prod_{k=1}^N \eta_k^\varepsilon(x_{T_k - \varepsilon}) \exp\left\{-\frac{\|\beta_{T_k - \varepsilon}^k(x_{T_k - \varepsilon})(L_k x_{T_k - \varepsilon} - v_k)\|^2}{2\varepsilon}\right\} \quad (13)$$

where for all $z \in \mathbb{R}^n$

$$\eta_k^\varepsilon(z) = \sqrt{\det(A_{T_k - \varepsilon}^k(z))} \quad \text{and} \quad C^\varepsilon = \prod_k \varepsilon^{-\frac{m_k}{2}}$$

Now using it in the case where $f = 1$, we get formally

$$\frac{\mathbb{E}[f(y^\varepsilon)\varphi^\varepsilon]}{\mathbb{E}[\varphi^\varepsilon]} = \frac{\mathbb{E}[f(x)\psi^\varepsilon]}{\mathbb{E}[\psi^\varepsilon]}$$

the fact that this quantity is finite is given by Proposition 1. The fact that the righthand term converges to the conditional expectation is given by Lemma 9 in the appendix. The proof relies essentially on the use of Aronson's estimates that provides gaussian bounds for transition probabilities.

The main difficulty of the proof consists in showing almost sure convergence and then uniform one for the φ^ε . An obvious candidate for the limit is

$$\varphi = \prod_{k=1}^N \varepsilon_k^{-\frac{m_k}{2}} \eta_k(y_{T_k}) \exp \left\{ -\frac{\|\beta_{T_k-\varepsilon_k}^k(y_{T_k-\varepsilon_k})(L_k y_{T_k-\varepsilon_k} - v_k)\|^2}{2\varepsilon_k} + \int_{T_k-\varepsilon_k}^{T_k} -\frac{(L_k y_t - v_k)^* A_t^k L_k b_t(y_t) ds}{T_k - t} \right. \\ \left. - \frac{(L_k y_t - v_k)^* d(A_t^k(y_t))(L_k y_t - v_k)}{2(T_k - t)} - \sum_{1 \leq i, j \leq m_k} \frac{d\langle A_{i,j}^k(y), (L_k y - v_k)_i (L_k y - v_k)_j \rangle_t}{2(T_k - t)} \right\} \quad (14)$$

Thanks to Lemma 10 given in the appendix, φ is well defined. As said before, we want to prove the following

Lemma 3. *There exists a decreasing sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ tending to 0 such that*

$$\lim_{i \rightarrow \infty} \mathbb{E} [\|\varphi^{\varepsilon_i} - \varphi\|] = 0$$

Proof. The proof is decomposed into two main parts. First one aims at showing the almost sure convergence of φ^{ε_i} . In second part we prove that $\mathbb{E}[\varphi^{\varepsilon_i}]$ tends to $\mathbb{E}[\varphi]$. Finally to conclude, we will use Scheffé's lemma.

For almost sure convergence, we first use triangular inequality

$$|\varphi^\varepsilon(y^\varepsilon) - \varphi(y)| \leq |\varphi^\varepsilon(y^\varepsilon) - \varphi^\varepsilon(y)| + |\varphi^\varepsilon(y) - \varphi(y)|$$

The second one converges to 0, this is given by Lemma 10. We now treat the term $|\varphi^\varepsilon(y^\varepsilon) - \varphi^\varepsilon(y)|$.

$$\frac{\varphi^\varepsilon(y^\varepsilon)}{\varphi^\varepsilon(y)} = \prod_{k=1}^N \frac{\eta_k^\varepsilon(y_{T_k-\varepsilon}^\varepsilon)}{\eta_k^\varepsilon(y_{T_k-\varepsilon})} \exp \left\{ -\frac{\|\beta_{T_k-\varepsilon_k}^k(y_{T_k-\varepsilon_k}^\varepsilon)(L_k y_{T_k-\varepsilon_k}^\varepsilon - v_k)\|^2}{2\varepsilon_k} - \frac{\|\beta_{T_k-\varepsilon_k}^k(y_{T_k-\varepsilon_k})(L_k y_{T_k-\varepsilon_k} - v_k)\|^2}{2\varepsilon_k} \right. \\ + \int_{T_k-\varepsilon_k}^{T_k-\varepsilon} -\frac{(L_k y_t^\varepsilon - v_k)^* A_t^k(y_t^\varepsilon) L_k b_t(y_t^\varepsilon) - (L_k y_t - v_k)^* A_t^k(y_t) L_k b_t(y_t)}{T_k - t} dt \\ - \frac{(L_k y_t^\varepsilon - v_k)^* d(A_t^k(y_t^\varepsilon))(L_k y_t^\varepsilon - v_k) - (L_k y_t - v_k)^* d(A_t^k(y_t))(L_k y_t - v_k)}{2(T_k - t)} \\ \left. - \sum_{i,j} \frac{d\langle A_{i,j}^k(y^\varepsilon), (L_k y^\varepsilon - v_k)_i (L_k y^\varepsilon - v_k)_j \rangle_t - d\langle A_{i,j}^k(y), (L_k y - v_k)_i (L_k y - v_k)_j \rangle_t}{2(T_k - t)} \right\}$$

We can write it respecting the order above

$$\frac{\varphi^\varepsilon(y^\varepsilon)}{\varphi^\varepsilon(y)} \stackrel{\text{Notation}}{=} \prod_{k=1}^N \Xi_k^\varepsilon \exp\{\Upsilon_k^\varepsilon + \Psi_k^\varepsilon + \Theta_k^\varepsilon + \Phi_k^\varepsilon\}$$

According to Lemma 2 there exists a decreasing sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ tending to 0 satisfying for all k that $y_{T_k-\varepsilon_i}^{\varepsilon_i}$ converges almost surely to y_{T_k} . From this we obtain the fact that $\Xi_k^{\varepsilon_i}$ converges almost surely to 1 and $\Upsilon_k^{\varepsilon_i}$ to 0 using regularity of σ . Then for all k

$$|\Psi_k^\varepsilon| \leq \int_{T_k-\varepsilon_k}^{T_k-\varepsilon} \left| \frac{L_k(y_t^\varepsilon - y_t)^* A_t^k(y_t^\varepsilon) L_k b_t(y_t^\varepsilon)}{T_k - t} \right| + \left| \frac{(L_k y_t - v_k)^* (A_t^k(y_t^\varepsilon) L_k b_t(y_t^\varepsilon) - A_t^k(y_t) L_k b_t(y_t))}{T_k - t} \right| dt$$

Since b and σ are bounded we use Lemma 1 to get

$$|\Psi_k^\varepsilon| \leq C \left(\int_{T_k-\varepsilon_k}^{T_k-\varepsilon} \frac{\|y_t^\varepsilon - y_t\|^2}{T_k - t} dt \right)^{\frac{1}{2}} \left(\int_{T_k-\varepsilon_k}^{T_k-\varepsilon} \frac{(1 + \log \log ((T_k - t)^{-1} + e))}{T_k - t} dt \right)^{\frac{1}{2}}$$

where C and C' are positive random variables. Thanks to Lemma 2, up to an extracted subsequence

$$\lim_{i \rightarrow \infty} \int_{T_k-\varepsilon_k}^{T_k-\varepsilon_i} \frac{\|y_t^{\varepsilon_i} - y_t\|^2}{T_k - t} dt = 0$$

that leads us to convergence for all k of $|\Psi_k^{\varepsilon_i}|$ to 0. Now we use Identity (34)

$$\begin{aligned}\Theta_k^\varepsilon = & \frac{\|Ly_t - v\|^2}{T-t} p_t(y_t) dt + \frac{\|Ly_t - v\|^2}{T-t} q_t(y_t) dw_t + \frac{\|Ly_t - v\|^2}{(T-t)^2} r_t(y_t) dt \\ & - \frac{\|Ly_t^\varepsilon - v\|^2}{T-t} p_t(y_t^\varepsilon) dt - \frac{\|Ly_t^\varepsilon - v\|^2}{T-t} q_t(y_t^\varepsilon) dw_t - \frac{\|Ly_t^\varepsilon - v\|^2}{(T-t)^2} r_t(y_t^\varepsilon) dt\end{aligned}$$

where p , q , and r are all $C_b^{1,2}$ functions. Hence using Lemmas 1 and 2 as above we obtain that $\lim_{\varepsilon_i \rightarrow 0} |\Theta_k^{\varepsilon_i}| = 0$ up to a subsequence. It remains to treat the term Φ_k^ε . Still using Identity (34), Lemmas 1 and 2 we show that $\lim_{\varepsilon_i \rightarrow 0} |\Phi_k^{\varepsilon_i}| = 0$ even if it means extracting once more a subsequence. We have obtained almost sure convergence of φ^ε to φ . Then we show the convergence of the expectations. For this we set a preliminary result

Proposition 1. *There exist two positive constants c_1 and c_2 such that for all $0 < \varepsilon < \varepsilon_0$*

$$c_1 \leq C^\varepsilon \mathbb{E}[\psi^\varepsilon] \leq c_2$$

Proof. We give an explicit expression

$$C^\varepsilon \mathbb{E}[\psi^\varepsilon] = \int q^\varepsilon(\zeta_1, \dots, \zeta_N) \prod_{k=1}^N \eta_k^\varepsilon(\zeta_k)^{-\frac{m_k}{2}} \exp\left\{-\frac{\|\beta_{T_k-\varepsilon}^k(\zeta_k)(L_k \zeta_k - v_k)\|^2}{2\varepsilon}\right\} d\zeta_k$$

where q^ε is the density of $(x_{T_1-\varepsilon}, \dots, x_{T_N-\varepsilon})$. Under theorem's assumptions x is a strong Markov process, with positive transition density. For $s, t \in [0, T]$, we denote $p_{s,t}(u, z)$ the density of $x_t^{s,u}$ solution of (1) initialized to be u at time s . Then thanks to Aronson's estimates there exist positive constant μ , λ , M and Λ such that the density p satisfies for $s < t$

$$\mu(t-s)^{-\frac{n}{2}} e^{-\frac{\lambda\|z-u\|^2}{t-s}} \leq p_{s,t}(u, z) \leq M(t-s)^{-\frac{n}{2}} e^{-\frac{\Lambda\|z-u\|^2}{t-s}}$$

Now using p we are able to write

$$q^\varepsilon(\zeta_1, \dots, \zeta_N) = p_{0, T_1-\varepsilon}(u, \zeta_1) \dots p_{T_{N-1}-\varepsilon, T_N-\varepsilon}(\zeta_{N-1}, \zeta_N)$$

Then we apply Aronson's estimates and the fact that for all i, j the coordinate $A_{i,j}^k$ is bounded by two positive constants. We obtain bounds for $C^\varepsilon \mathbb{E}[\psi^\varepsilon]$ of the type

$$\lambda^{-1} C^\varepsilon \int \exp\left\{\sum_{j=1}^N \frac{-\lambda\|L_k \zeta_k - v_k\|^2}{2\varepsilon} - \frac{\lambda\|\zeta_1 - u\|^2}{T_1 - \varepsilon} - \sum_{k=2}^N \frac{\lambda\|\zeta_k - \zeta_{k-1}\|^2}{T_k - T_{k-1}}\right\} \prod_{k=1}^N d\zeta_k$$

where λ is a positive constant large for the lower bound and small for the upper one. The integral can be interpreted as a gaussian expectation where

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 - \zeta_1 \\ \vdots \\ \zeta_N - \zeta_{N-1} \end{pmatrix}$$

is a centered gaussian vector with covariance matrix

$$R^\varepsilon = \frac{1}{2\lambda} \begin{pmatrix} (T_1 - \varepsilon)I_n & 0 & \dots & 0 \\ 0 & (T_2 - T_1)I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & (T_N - T_{N-1})I_n \end{pmatrix}$$

where I_n is the n -dimensional identity matrix. As a remark, in the sense of symmetric matrices, there exist two positive constants c_1 and c_2 such that

$$c_1 I_{Nn} < R^\varepsilon < c_2 I_{Nn}$$

where I_{Nn} is the Nn -dimensional identity matrix. Thus the gaussian vector

$$\begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_N \end{pmatrix}$$

admits for covariance matrix

$$\Gamma^\varepsilon = G^{-1} R^\varepsilon G^{-*}$$

where

$$G = \begin{pmatrix} I_n & 0 & \dots & \dots & 0 \\ -I_n & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -I_n & I_n \end{pmatrix}$$

We still keep bounds for the covariance matrix

$$c_1 I_{Nn} < \Gamma^\varepsilon < c_2 I_{Nn}$$

Now we can get bounds for $C^\varepsilon \mathbb{E}[\psi^\varepsilon]$ with expectations of type

$$\lambda^{\frac{m_k}{2}} C^\varepsilon \mathbb{E}[\exp\{-\sum_{k=1}^N \frac{\lambda \|L_k X_k - v_k\|^2}{2\varepsilon}\}]$$

where X_k is a n -dimensional gaussian variable. Then we use Lemma 11 given in the appendix to obtain the fact that

$$\begin{aligned} \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_N} &\rightarrow \mathbb{R} \\ (v_1, \dots, v_N) &\mapsto C^\varepsilon \prod_{k=1}^N \lambda^{\frac{m_k}{2}} \mathbb{E}[\exp\{-\frac{\lambda \|L_k X_k - v_k\|^2}{2\varepsilon}\}] \end{aligned}$$

is a gaussian density of a variable $(L_1 X_1 + \sqrt{\frac{\varepsilon}{\lambda}} Y_1, \dots, L_N X_N + \sqrt{\frac{\varepsilon}{\lambda}} Y_N)$ where the Y_k are m_k -dimensional centered normalized gaussian vectors. Moreover the two families $(X_k)_k$ and $(Y_k)_k$ are independant. Finally using bounds obtained above for Γ^ε we get the fact that for all $0 < \varepsilon < \varepsilon_0$

$$c_1 < C^\varepsilon \mathbb{E}[\psi^\varepsilon] < c_2$$

□

As a first consequence, thanks to identity (11), $\mathbb{E}[\varphi^\varepsilon]$ is finite so that $\frac{\varphi^\varepsilon}{\mathbb{E}[\varphi^\varepsilon]}$ is a density. We may also use Fatou's lemma to get

$$\mathbb{E}[\varphi] \leq \liminf_{\varepsilon \rightarrow 0} \mathbb{E}[\varphi^\varepsilon] \leq c_2$$

It takes more work to control $\limsup_{\varepsilon \rightarrow 0} \mathbb{E}[\varphi^\varepsilon]$.

Let $J > 0$ be a large number, we introduce for all process $(z_t)_{t \in [0, T]}$ and for all $1 \leq k \leq N$ the stopping time τ_k^ε

$$\begin{aligned} \tau_k^\varepsilon &= \inf\{t_k < t \leq T_k - \varepsilon : \frac{1}{\sqrt{T_k - t}} \exp\{-\frac{\|L_k z_t - v_k\|^2}{2(T_k - t)} D\} \leq J^{-1}\} \\ &= \inf\{t_k < t \leq T_k - \varepsilon : \frac{\|L_k z_t - v_k\|^2}{2(T_k - t)} \geq D^{-1} \log\left(\frac{J}{\sqrt{T_k - t}}\right)\} \end{aligned}$$

where D is a positive constant such that $D I_d \leq A_k$. We know that such a constant exists according to assumptions on the function a . The t_k are chosen to be real numbers contained in $(T_{k-1}, T_k - \varepsilon)$. As a convention we set $\tau_k^\varepsilon = T_k$ if the condition is empty. Let τ^ε be the first of the τ_k^ε such that the condition is non-empty

$$\tau^\varepsilon = \inf_k \{\tau_k^\varepsilon : \tau_k^\varepsilon < T_k\}$$

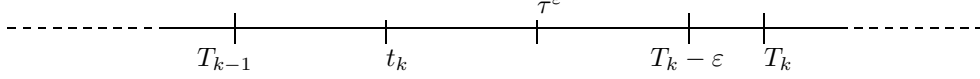
we set as convention $\tau^\varepsilon = T$ if for all k , $\tau_k^\varepsilon = T_k$. Even if it means changing f in Equation (11)

$$\frac{\mathbb{E}[f(y^\varepsilon)\mathbf{1}_{\tau^\varepsilon < T} \varphi^\varepsilon]}{\mathbb{E}[\varphi^\varepsilon]} = \frac{\mathbb{E}[f(x)\mathbf{1}_{\tau^\varepsilon < T} C^\varepsilon \psi^\varepsilon]}{\mathbb{E}[C^\varepsilon \psi^\varepsilon]}$$

We recall that

$$C^\varepsilon \psi^\varepsilon = \prod_k \varepsilon^{-\frac{m_k}{2}} \exp\left\{-\frac{\|\beta_{T_k-\varepsilon}^k(L_k x_{T_k-\varepsilon} - v_k)\|^2}{2\varepsilon}\right\}$$

We now consider to be on set $\{\tau^\varepsilon = \tau_k^\varepsilon\}$



We decompose $C^\varepsilon \psi^\varepsilon$ into three parts as a product of three factors

$$\begin{aligned} F_1 &= \prod_{j < k} \varepsilon^{-\frac{m_j}{2}} \exp\left\{-\frac{\|\beta_{T_j-\varepsilon}^j(L_j x_{T_j-\varepsilon} - v_j)\|^2}{2\varepsilon}\right\} \\ F_2 &= \varepsilon^{-\frac{m_k}{2}} \exp\left\{-\frac{\|\beta_{T_k-\varepsilon}^k(L_k x_{T_k-\varepsilon} - v_k)\|^2}{2\varepsilon}\right\} \\ F_3 &= \prod_{j > k} \varepsilon^{-\frac{m_j}{2}} \exp\left\{-\frac{\|\beta_{T_j-\varepsilon}^j(L_j x_{T_j-\varepsilon} - v_j)\|^2}{2\varepsilon}\right\} \end{aligned}$$

We are interested in

$$\mathbb{E}[C^\varepsilon \psi^\varepsilon \mathbf{1}_{\tau^\varepsilon = \tau_k^\varepsilon}] = \mathbb{E}[F_1 F_2 F_3 \mathbf{1}_{\tau^\varepsilon = \tau_k^\varepsilon}]$$

We now use Markov's property to get independence between Past and Future knowing Present (cf [3] see last chapter about conditional expectations)

$$\mathbb{E}[C^\varepsilon \psi^\varepsilon \mathbf{1}_{\tau^\varepsilon = \tau_k^\varepsilon}] = \mathbb{E}[F_1 \mathbb{E}[F_2 F_3 \mathbf{1}_{\tau^\varepsilon = \tau_k^\varepsilon} | \tau_k^\varepsilon, x_{\tau_k^\varepsilon}]] = \mathbb{E}[F_1 \mathbb{E}[F_2 \mathbf{1}_{\tau^\varepsilon = \tau_k^\varepsilon} | \tau_k^\varepsilon, x_{\tau_k^\varepsilon}] \mathbb{E}[F_3 | x_{\tau_k^\varepsilon}]] \quad (15)$$

$$= \mathbb{E}[F_1 \mathbb{E}[F_2 \mathbf{1}_{\tau^\varepsilon = \tau_k^\varepsilon} | \tau_k^\varepsilon, x_{\tau_k^\varepsilon}] \mathbb{E}[F_3 | x_{T_k-\varepsilon}]] \quad (16)$$

In order to study the factor $\mathbb{E}[F_2 \mathbf{1}_{\tau^\varepsilon = \tau_k^\varepsilon} | \mathcal{F}_{\tau_k^\varepsilon}]$ we introduce

$$\theta_t = \frac{1}{\sqrt{T_k - t}} \exp\left\{-\frac{\|\beta_t^k(L_k x_t - v_k)\|^2}{2(T_k - t)}\right\}$$

For $t \in [T_{k-1}, T_k - \varepsilon]$, we set $z_t = L_k x_t - v_k$, $p_t = \|\beta_t^k(L_k x_t - v_k)\|$. We recall that $\beta_t^k = \sigma_t^* L_k^* A_t^k$ with $A_t^k = (\beta_t^k)^* \beta_t^k = (L_k a_t L_k^*)^{-1}$. With respect to these notations, we have

$$p_t^2 = z_t^* A_t^k z_t$$

It is also easy to see that

$$dz_t = L_k b_t dt + L_k \sigma_t dw_t$$

and then $d\langle z \rangle_t = L_k a_t L_k^* dt = (A_t^k)^{-1} dt$. We use Itô's formula

$$d(p_t^2) = d(z_t^* A_t^k z_t) = 2z_t^* A_t^k dz_t + z_t^* dA_t^k z_t + \sum_{i,j} d\langle A_{i,j}^k, z_i z_j \rangle_t + m_k dt$$

Then

$$d\frac{p_t^2}{T_k - t} = \frac{2z_t^* A_t^k dz_t}{T_k - t} + \frac{p_t^2 dt}{(T_k - t)^2} + \frac{z_t^* dA_t^k z_t}{T_k - t} + \frac{m_k dt}{T_k - t} + \frac{\sum_{i,j} d\langle \Delta_{i,j}, z_i z_j \rangle_t}{T_k - t}$$

First using definitions of z , β^k and A^k we get

$$\begin{aligned} z_t^* A_t^k dz_t &= z_t^* A_t^k L_k dx_t = z_s^* A_t^k L_k \sigma_t \sigma_t^{-1} dx_t \\ &= z_t^* (\beta_t^k)^* \sigma_t^{-1} b_t dt + z_t^* (\beta_t^k)^* dw_t \end{aligned}$$

This leads us to the existence of two bounded adapted processes $r^{(1)}$ and $r^{(2)}$ defined on $[T_{k-1}, T_k - \varepsilon]$ such that

$$z_t^* A_t^k dz_t = p_t r_t^{(1)} dt + p_t r_t^{(2)} dw_t$$

In a same way we remark that there exist two bounded adapted processes $r^{(3)}$ and $r^{(4)}$ such that

$$d(L_k a_t L_k^*) = r_t^{(3)} dt + r_t^{(4)} dw_t$$

we get even if it means changing $r^{(3)}$ and $r^{(4)}$

$$z_t^* dA_t^k z_t = z_t^* (d(L_k a_t L_k^*)^{-1}) z_t = p_t^2 r_t^{(3)} dt + p_t^2 r_t^{(4)} dw_t$$

Finally, we obtain existence of two bounded adapted processes r and r' such that

$$d \frac{p_t^2}{T-t} = \frac{2p_t dw_t}{T-t} + \frac{p_t^2 dt}{(T-t)^2} + r_t \frac{p_t^2}{T-t} dw_t + \frac{dt}{T-t} + r'_t \frac{p_t^2 + p_t}{T-t} dt$$

From this we deduce that quadratic variation

$$d \left\langle \frac{p_t^2}{T-t} \right\rangle = \frac{4p_t^2 + r_t^2 p_t^4 + 4r_t p_t^3}{(T-t)^2} dt$$

Now we apply Itô's formula to the function θ always for $t \in [T_{k-1}, T_k - \varepsilon]$

$$d\theta_t = \frac{\theta_t dt}{2(T_k - t)} - \frac{1}{2} \theta_t d \left(\frac{p_t^2}{T_k - t} \right) + \frac{1}{8} \theta_t d \left\langle \frac{p_t^2}{T_k - t} \right\rangle$$

We deduce from the three last equations after simplification of four terms that there exists a martingale M and a bounded adapted process r'' both defined on $[T_{k-1}, T_k - \varepsilon]$ such that

$$d\theta_t = dM_t + \theta_t r_t'' \left(\frac{p_t^2 + p_t}{T_k - t} + \frac{p_t^4 + p_t^3}{(T_k - t)^2} \right) dt$$

For any $\eta > 0$, functions $x \mapsto e^{-\eta \frac{x^2}{2}} |z|^m$ for $m = 1, 2, 3, 4$ are all bounded, then there exists a constant c_η such that

$$(\sqrt{T_k - t} \theta_t)^\eta \left(\frac{p_t^2 + p_t}{T_k - t} + \frac{p_t^4 + p_t^3}{(T_k - t)^2} \right) \leq \frac{c_\eta}{\sqrt{T_k - t}}$$

This gives us the existence of a bounded adapted process π defined on $[T_{k-1}, T_k - \varepsilon]$ that allows us to write

$$d\theta_t = dM_t + \theta_t^{1-\eta} (T_k - t)^{-h} \alpha_t dt$$

with $h = \frac{1+\eta}{2}$. We now integrate it for $t \in (\tau_k^\varepsilon, T_k - \varepsilon]$

$$\theta_t = \theta_{\tau_k^\varepsilon} + M_t - M_{\tau_k^\varepsilon} + \int_{\tau_k^\varepsilon}^t \pi_s \theta_s^{1-\eta} (T_k - s)^{-h} ds$$

This leads to the following

$$\mathbb{E}[\theta_t \mathbf{1}_{\tau_k^\varepsilon < t}] \leq J^{-1} + \bar{\pi} \int_{t_k}^t \mathbb{E}[\theta_s \mathbf{1}_{\tau_k^\varepsilon < s}]^{1-\eta} (T_k - s)^{-h} ds$$

where $\bar{\pi} = \sup_s |\pi_s|$. So $\mathbb{E}[\theta \mathbf{1}_{\tau_k^\varepsilon < t}]$ is bounded by the solution u of

$$du_s = \bar{\alpha} u_s^{1-\eta} (T_k - s)^{-h} ds, \quad u_{t_k} = J^{-1}$$

and this equation has an explicit solution

$$u_t = \left\{ \frac{\eta \bar{\alpha}}{1-\eta} [(T_k - t_k)^{1-h} - (T_k - t)^{1-h}] + J^{-\eta} \right\}^{\frac{1}{\eta}} \leq \{c_k (T_k - t_k)^{1-h} + J^{-\eta}\}^{\frac{1}{\eta}}$$

where c_k is a positive constant. Then for all $t \in [T_{k-1} - T_k - \varepsilon]$

$$\mathbb{E}[\theta_t \mathbf{1}_{t > \tau_k^\varepsilon} | \tau_k^\varepsilon, x_{\tau_k^\varepsilon}] \leq \{c_k(T_k - t_k)^{1-h} + J^{-\eta}\}^\eta$$

In particular when $t = T_k - \varepsilon$

$$\mathbb{E}[F_2 \mathbf{1}_{t > \tau_k^\varepsilon} | \tau_k^\varepsilon, x_{\tau_k^\varepsilon}] = \mathbb{E}[\theta_{T_k - \varepsilon} \mathbf{1}_{t > \tau_k^\varepsilon} | \tau_k^\varepsilon, x_{\tau_k^\varepsilon}] \leq \{c_k(T_k - t_k)^{1-h} + J^{-\eta}\}^\eta$$

We now come back to equation (15), we get a first bound

$$\mathbb{E}[C^\varepsilon \psi^\varepsilon \mathbf{1}_{\tau^\varepsilon = \tau_k^\varepsilon}] \leq \{c_k(T_k - t_k)^{1-h} + J^{-\eta}\}^\eta \mathbb{E}[F_1 \mathbb{E}[F_3 | x_{T_k - \varepsilon}]]$$

In order to treat the factor $\mathbb{E}[F_3 | x_{T_k - \varepsilon}]$ we use Aronson's estimate to get

$$\mathbb{E}[F_3 | \mathcal{F}_{T_k - \varepsilon}] \leq G \int \prod_{j > k} \frac{1}{\sqrt{\varepsilon}} \exp\left\{-\frac{\|\beta_{T_j - \varepsilon}^j(L_j \zeta_j - v_j)\|^2}{2\varepsilon}\right\} \frac{1}{T_j - T_{j-1}} \exp\left\{-\Lambda_j \frac{|\zeta_j - \zeta_{j-1}|^2}{T_j - T_{j-1}}\right\} d\zeta_j$$

where G is a positive constant. We just have to use Lemma 11 given in the appendix to obtain an positive constant upper bound. The same Lemma 11 brings us a positive constant upper bound for $\mathbb{E}[F_1]$. Finally the inequation we get from equation (15) is the following

$$\mathbb{E}[C^\varepsilon \psi^\varepsilon \mathbf{1}_{\tau^\varepsilon = \tau_k^\varepsilon}] \leq G \{c_k(T_k - t_k)^{1-h} + J^{-\eta}\}^\eta$$

where G is a positive constant. From this we deduce

$$\mathbb{E}[C^\varepsilon \psi^\varepsilon \mathbf{1}_{T > \tau^\varepsilon}] = \sum_k \mathbb{E}[C^\varepsilon \psi^\varepsilon \mathbf{1}_{\tau^\varepsilon = \tau_k^\varepsilon}] \leq G \max_k \{(c_k(T_k - t_k)^{1-h} + J^{-\eta})^\eta\}$$

According to this last result and using the lower bound of $C^\varepsilon \mathbb{E}[\psi^\varepsilon]$ given by Proposition 1 we finally have

$$\frac{\mathbb{E}[C^\varepsilon \psi^\varepsilon \mathbf{1}_{T = \tau^\varepsilon}]}{\mathbb{E}[C^\varepsilon \psi^\varepsilon]} \geq 1 - G \max_k \{(c_k(T_k - t_k)^{1-h} + J^{-\eta})^\eta\}$$

where G is positive constant. So using inequality (11) we obtain

$$\frac{\mathbb{E}[\varphi^\varepsilon \mathbf{1}_{T = \tau^\varepsilon}]}{\mathbb{E}[\varphi^\varepsilon]} \geq 1 - G \max_k \{(c_k(T_k - t_k)^{1-h} + J^{-\eta})^\eta\} \quad (17)$$

Moreover the family $(\varphi^\varepsilon \mathbf{1}_{T = \tau^\varepsilon})_\varepsilon$ is uniformly integrable. Indeed by definition of τ^ε we can get upper bounds depending on J for the different factors in Expression (12) of φ^ε or (14) of φ , for all $0 \leq \varepsilon < 1$

$$\begin{aligned} \varphi^\varepsilon = & \prod_{k=1}^N \varepsilon_k^{-\frac{m_k}{2}} \eta_k^\varepsilon(y_{T_k - \varepsilon}^\varepsilon) \exp\left\{-\int_{T_k - \varepsilon_k}^{T_k - \varepsilon} \frac{(L_k y_t^\varepsilon - v_k)^* A_t^k L_k b_t dt}{T_k - t} - \frac{(L_k y_t^\varepsilon - v_k)^* dA_t^k (L_k y_t^\varepsilon - v_k)}{2(T_k - t)}\right. \\ & \left.- \sum_{1 \leq i, j \leq m_k} \frac{d\langle A_{i,j}^k, (L_k y^\varepsilon - v_k)_i (L_k y^\varepsilon - v_k)_j \rangle_t}{2(T_k - t)} - \frac{\|\beta_{T_k - \varepsilon_k}^k(y_{T_k - \varepsilon_k}^\varepsilon)(L_k y_{T_k - \varepsilon_k}^\varepsilon - v_k)\|^2}{2\varepsilon_k}\right\} \end{aligned}$$

in fact " $\varphi^0 = \varphi$ ". We recall that b and σ are bounded so is η . Then on $\{\tau^\varepsilon = T\}$

$$\left\| \frac{(L_k y_t^\varepsilon - v_k)^* A_t^k L_k b_t}{T_k - t} \right\| \leq \frac{C}{\sqrt{T_k - t}} \sqrt{\log \left(\frac{J}{(T_k - t)^{\frac{m_k}{2}}} \right)}$$

which is an integrable quantity in T_k , and C is a positive constant depending on the choice of b and σ . A same method gives an upper bound for the terms where quadratic variation appears. For the terms in dA_t^k , we decompose with respect to integrals with respect to dt and dw_t

$$\frac{(L_k y_t^\varepsilon - v_k)^* dA_t^k (L_k y_t^\varepsilon - v_k)}{2(T_k - t)} = \frac{\|L_k^* y_t^\varepsilon - v_k\|^2}{T_k - t} r_t^k(y_t^\varepsilon) dw_t + \frac{\|L_k y_t^\varepsilon - v_k\|^2}{T_k - t} q_t^k(y_t^\varepsilon) dt$$

where r^k and q^k are bounded adapted functions. Then for fixed J , there exists a constant K such that

$$\varphi^\varepsilon \mathbf{1}_{T = \tau^\varepsilon} \leq C \prod_k \exp\left\{\int_{T_k - \varepsilon_k}^{T_k} \frac{\|L_k y_t - v_k\|^2}{T_k - t} r_t^k dw_t - \frac{1}{2} \frac{\|L_k y_t - v_k\|^4}{(T_k - t)^2} \|r_t^k\|^2 dt\right\}$$

where C is a positive constant. Let us recall the following lemma (cf [6] p.198)

Lemma 4 (Novikov). *Let $(M_t)_{t \in \mathbb{R}}$ be a continuous local martingale, we set for all t*

$$Z_t = \exp\{M_t - \frac{1}{2}\langle M \rangle_t\}$$

If

$$\mathbb{E}[\exp\{\frac{1}{2}\langle M \rangle_t\}] < +\infty$$

then we have

$$\mathbb{E}[Z_t] = 1$$

Let us remark that for all $p > 0$

$$\exp\{p \sum_k \int_{T_k - \varepsilon_k}^{T_k - \varepsilon} \frac{\|L_k y_t^\varepsilon - v_k\|^4}{2|T_k - t|^2} \|r_t^k\|^2 dt\} \leq \exp\{p \sum_k \int_{T_k - \varepsilon_k}^{T_k - \varepsilon} \left[\log \left(\frac{J}{(T_k - t)^{\frac{m_k}{2}}} \right) \right]^2 dt\} \leq C^p$$

where C is a positive constant. Thus, we apply Novikov's lemma to get uniform integrability. Then we take the $\liminf_{\varepsilon \rightarrow 0}$ and use Lebesgue's theorem to obtain

$$\frac{\mathbb{E}[\varphi \mathbf{1}_{T=\tau^\varepsilon}]}{\limsup_{\varepsilon \rightarrow 0} \mathbb{E}[\varphi^\varepsilon]} \geq 1 - N \max_k \{(c_k(T_k - \varepsilon - t_k)^{1-h} + J^{-\eta})\}^{\frac{1}{\eta}} \quad (18)$$

Now $\mathbf{1}_{T=\tau^\varepsilon}$ converges almost surely to 1 as J tends to infinity. We are able to say after making the t_k tend to T_k that

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{E}[\varphi^\varepsilon] \leq \mathbb{E}[\varphi] \quad (19)$$

We finish the proof by Scheffé's Lemma (cf [3] p.36)

Lemma 5 (Scheffé). *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of positive functions converging to f , moreover we suppose that*

$$\lim_{n \rightarrow \infty} \mathbb{E}[f_n] = \mathbb{E}[f] < \infty$$

then the sequence (f_n) converges to f in \mathbb{L}^1 .

□

Finally we conclude thanks to Lemmas 9 and 3.

□

Case where b is unbounded

Suppose now that b is locally Lipschitz with respect to x and is locally bounded. Moreover the SDE (1) admits a strong solution. We use a Girsanov theorem to reduce the problem to the case of a bounded drift.

We recall the Girsanov theorem for unbounded drifts introduced in [4]

Theorem 2. *Let b , h and σ be measurable functions from $\mathbb{R}^+ \times \mathbb{R}^n$ to \mathbb{R}^n , \mathbb{R}^d and $\mathbb{R}^{n \times d}$ locally Lipschitz with respect to x ; consider the following SDE's*

$$\begin{aligned} dx_t &= b_t(x_t)dt + \sigma_t(x_t)dw_t, \\ dy_t &= (b_t(y_t) + \sigma_t(y_t)h_t(y_t))dt + \sigma_t(y_t)d\tilde{w}_t, \\ x_0 &= y_0 \end{aligned}$$

on the finite interval $[0, T]$. We assume the existence of strong solution for each equation. We assume in addition that h is bounded on compact sets. Then the Girsanov formula holds: for any non negative Borel function f defined on $C([0, T], \mathbb{R}^n)$, one has

$$\begin{aligned} \mathbb{E}[f(y, \tilde{w}^h)] &= \mathbb{E}[f(x, w) \exp\{\int_0^T h_t^*(x_t)dw_t - \frac{1}{2} \int_0^T \|h_t(x_t)\|^2 dt\}] \\ \mathbb{E}[f(x, w)] &= \mathbb{E}[f(y, \tilde{w}^h) \exp\{-\int_0^T h_t^*(y_t)d\tilde{w}_t - \frac{1}{2} \int_0^T \|h_t(y_t)\|^2 dt\}] \end{aligned}$$

where $\tilde{w}^h = \tilde{w}_t + \int_0^t h_s(y_s)ds$.

Thanks to both Theorems 1 and 2 we obtain

Theorem 3. Suppose σ and a^{-1} to be $C_b^{1,2}$ -functions. Assume that b is a locally Lipschitz with respect to x and locally bounded function. Let y be the solution of

$$dy_t = \hat{b}_t(y_t)dt + \sigma_t(y_t)d\tilde{w}_t - \sum_{k=1}^N \sigma_t(y_t)\beta_t^k(y_t) \frac{L_k y_t - v_k}{T_k - t} \mathbf{1}_{(T_k - \varepsilon_k, T_k)}(t)dt$$

where \hat{b} satisfies the assumptions of Theorem 1.

Then for any bounded continuous function f

$$\begin{aligned} & \mathbb{E}[f(x)|(L_k x_{T_k} = v_k)_{1 \leq k \leq N}] \\ &= C \mathbb{E} \left[f(y) \prod_{k=1}^N \eta_k(y_{T_k}) \exp \left\{ - \frac{\|\beta_{T_k - \varepsilon_k}^k(L_k y_{T_k - \varepsilon_k} - v_k)\|^2}{2\varepsilon_k} + \int_{T_k - \varepsilon_k}^{T_k} - \frac{(L_k y_s - v_k)^* L_k \hat{b}_s(y_s) ds}{T_k - s} \right. \right. \\ & \quad - \frac{(L_k y_s - v_k)^* d(A_t^k(y_t)) (L_k y_s - v_k)}{2(T_k - s)} - \sum_{1 \leq i, j \leq m_k} \frac{d\langle A^k(y)_{i,j}, (L_k y_{\cdot} - v_k)_i (L_k y_{\cdot} - v_k)_j \rangle_s}{2(T_k - s)} \\ & \quad \left. \left. + \int_0^T \check{b}_t^*(y_t) a_t(y_t)^{-1} dy_t - \frac{1}{2} \|\sigma_t(y_t)^{-1} \check{b}_t(y_t)\|^2 dt \right\} \right] \end{aligned}$$

where C is a positive constant and $\check{b} = b - \hat{b}$.

Proof. Let \hat{x} be the solution of

$$d\hat{x}_t = \hat{b}_t(\hat{x}_t)dt + \sigma_t(\hat{x}_t)dw_t, \quad \hat{x}_0 = u$$

Then from Theorem 2, for any bounded continuous function f and g

$$\begin{aligned} & \mathbb{E}[f(x)g(L_1 x_{T_1}, \dots, L_N x_{T_N})] = \mathbb{E}[f(\hat{x}_t)g(L_1 \hat{x}_{T_1}, \dots, L_N \hat{x}_{T_N}) e^{\int_0^T \check{b}_t^*(\hat{x}_t) a_t(\hat{x}_t)^{-1} d\hat{x}_t - \frac{1}{2} \|\sigma_t(\hat{x}_t)^{-1} \check{b}_t(y_t)\|^2 dt}] \\ &= \int \mathbb{E}[f(\hat{x}_t) e^{\int_0^T \check{b}_t^*(\hat{x}_t) a_t(\hat{x}_t)^{-1} d\hat{x}_t - \frac{1}{2} \|\sigma_t(\hat{x}_t)^{-1} \check{b}_t(y_t)\|^2 dt} | (L_k \hat{x}_{T_k} = v_k)_{1 \leq k \leq N}] g(v_1, \dots, v_N) \prod_k dv_k \end{aligned}$$

It remains to apply Theorem 1. □

Appendix

Lemma 6. Let us consider Equation (3) with random initial condition u on $[0, T]$ with $N = 1$ which means only one observation time in T .

$$dy_t = b_t(y_t)dt + \sigma_t(y_t)\tilde{w}_t - P_t(y_t) \frac{y_t - u_1}{T - t} \mathbf{1}_{(T - \varepsilon_1, T)}(t), \quad y_0 = u$$

Then this equation admits a unique solution on $[0, T)$. Moreover $\|L(y_t - u_1)\|^2 \leq C(\omega)(T - t) \log \log[(T - t)^{-1} + e]$ a.s., where C is a positive random variable.

Proof. We recall that parameters b and σ are locally Lipschitz functions. So that the equation admits a unique solution on both intervals $[0, T - \varepsilon_1]$ and $(T - \varepsilon_1, T)$ and so on $[0, T)$. Moreover thanks to Itô's formula, on $(T - \varepsilon_1, T)$

$$d \frac{L(y_t - u_1)}{T - t} = (T - t)^{-1} L[b_t dt + \sigma_t d\tilde{w}_t - P_t \frac{y_t - u_1}{T - t} dt] + L \frac{y_t - u_1}{(T - t)^2} dt$$

then using (4), we have $LP_t = L$ so that

$$d \frac{L(y_t - u_1)}{T - t} = (T - t)^{-1} L[b_t dt + \sigma_t d\tilde{w}_t]$$

For all $1 \leq i \leq n$ the process $\{(\int_0^t (T-s)^{-1} \sigma_s(y_s) d\tilde{w}_s)_i, t \geq 0\}$ is a continuous local martingale whose quadratic variation $\tau_t = \int_0^t \sum_{j=1}^n (T-s)^{-2} \sigma_s(y_s)_{i,j} ds$ satisfies $\lim_{t \rightarrow T} \tau_t = +\infty$ and $\tau_t \leq \frac{c}{T-t}$ where c is a positive constant. Hence we just have to apply the Dambis-Dubins-Schwarz theorem that gives us the existence of a Brownian motion B^i such that

$$\left(\int_0^t (T-s)^{-1} \sigma_s(y_s) d\tilde{w}_s \right)_i = B^i(\tau_t)$$

The law of iterated logarithm allows us to conclude. \square

Lemma 7. *Let us consider Equation (3) with random initial condition u on $[0, T]$ with $N = 1$ which means only one observation time in T*

$$dy_t = b_t(y_t)dt - P_t(y_t) \frac{y_t - u_1}{T-t} dt, \quad y_0 = u$$

Then for all $s < t < T$,

$$\frac{\mathbb{E}[\|L(y_t - u_1)\|^2]}{T-t} \leq c(1 + \sqrt{T-t} \mathbb{E}[\|L(u - u_1)\|^2]) \quad (20)$$

and

$$\mathbb{E}[\|y_s - y_t\|^2] \leq C(t-s)(1 + \sqrt{T-s} \mathbb{E}[\|L(u - u_1)\|^2]) \quad (21)$$

where c and C are positive constants depending on T , ε_1 , bounds for b and σ .

Proof. Thanks to Identity (4), on $(T - \varepsilon_1, T)$

$$dL(y_t - u_1) = L[b_t dt + \sigma_t d\tilde{w}_t] - L \frac{y_t - u_1}{T-t} dt$$

Thus

$$d(\|L(y_t - u_1)\|^2) = 2(y_t - u_1)^* L^* L[b_t dt + \sigma_t d\tilde{w}_t] - 2 \frac{\|L(y_t - u_1)\|^2}{T-t} dt + \text{Tr}(La_t L^*) dt$$

where the function Tr gives the sum of all diagonal terms. Finally

$$d\left(\frac{\|L(y_t - u_1)\|^2}{T-t}\right) = 2 \frac{(y_t - u_1)^*}{T-t} L^* L[b_t dt + \sigma_t d\tilde{w}_t] + \frac{\text{Tr}(La_t L^*)}{T-t} dt - \frac{\|L(y_t - u_1)\|^2}{(T-t)^2} dt$$

Setting $E_t = \mathbb{E}\left[\frac{\|L(y_t - u_1)\|^2}{T-t}\right]$, since b and σ are bounded, we get

$$E'_t \leq C_1 \left(\frac{\sqrt{E_t} + 1}{T-t} \right) - \frac{E_t}{T-t} \quad (22)$$

where C_1 is a positive constant depending on $\|b\|_\infty$ and $\|\sigma\|_\infty$.

$$E'_t \leq (T-t)^{-1} \left[C_1 \left(\frac{E_t}{2C_1} + \frac{C_1}{2} + 1 \right) - E_t \right] = (T-t)^{-1} (C - \frac{E_t}{2}) \quad (23)$$

where $C = C_1 + \frac{C_1^2}{2}$. Thus

$$\left(\frac{E_t - 2C}{\sqrt{T-t}} \right)' = \frac{E'_t}{\sqrt{T-t}} + \frac{E_t - 2C}{2(T-t)^{\frac{3}{2}}} \leq 0$$

thanks to (23). Hence

$$\frac{E_t - 2C}{\sqrt{T-t}} \leq \frac{E_{T-\varepsilon_1} - 2C}{\sqrt{\varepsilon_1}}$$

that can be written

$$E_t \leq 2C + \sqrt{\frac{T-t}{\varepsilon_1}} (E_{T-\varepsilon_1} - 2C)$$

Similarly for $t < T - \varepsilon_1$, Inequality 22 becomes

$$E_t \leq C'(E_0 + 1) \exp\left\{\frac{C't}{\varepsilon_1}\right\}$$

where C' is a positive constant only depending on T and bounds of b and σ . That gives us (20).

By definition for $s, t \in (T - \varepsilon_1, T)$ we have

$$y_s - y_t = \int_s^t b_\tau d\tau + \sigma_\tau d\tilde{w}_\tau - P_\tau \frac{y_\tau - u_1}{T - \tau} d\tau$$

Since b and σ are bounded functions, using Minkovski's inequality

$$\mathbb{E}[\|y_s - y_t\|^2]^{\frac{1}{2}} \leq \mathbb{E}[\|\int_s^t b_\tau d\tau\|^2]^{\frac{1}{2}} + \mathbb{E}[\|\int_s^t \sigma_\tau d\tilde{w}_\tau\|^2]^{\frac{1}{2}} + \mathbb{E}[\|\int_s^t P_\tau \frac{y_\tau - u_1}{T - \tau} d\tau\|^2]^{\frac{1}{2}} \quad (24)$$

Thanks to Doob's inequality (see *e.g.* [5] p.170) we get

$$\mathbb{E}[\|\int_s^t b_\tau d\tau\|^2] + \mathbb{E}[\|\int_s^t \sigma_\tau d\tilde{w}_\tau\|^2] \leq C_2(t - s)$$

where $C_2 = C_1^2$ is the square of the constant introduced above. In order to treat the last term in (24), we beforehand give a property

Proposition 2. *Let f be a real-valued process defined on a segment $[a, b]$, then*

$$\mathbb{E}[(\int_a^b f_s ds)^2]^{\frac{1}{2}} \leq \int_a^b \mathbb{E}[f_s^2]^{\frac{1}{2}} ds$$

Proof. Indeed

$$\mathbb{E}[(\int_a^b f_s ds)^2] \leq \mathbb{E}[(\int_a^b |f_s| ds)^2] = 2\mathbb{E}[\int_a^b \int_a^s |f_u| du |f_s| ds] = 2 \int_a^b \int_a^s \mathbb{E}[|f_u f_s|] du ds$$

so that

$$\mathbb{E}[(\int_a^b f_s ds)^2] \leq 2 \int_a^b \int_a^s \mathbb{E}[(f_u)^2]^{\frac{1}{2}} \mathbb{E}[(f_s)^2]^{\frac{1}{2}} du ds = (\int_a^b \mathbb{E}[f_s^2]^{\frac{1}{2}} ds)^2$$

□

Thanks to Proposition 2, assumptions (4) on matrix P and result (20)

$$\begin{aligned} \mathbb{E} \left[\left\| \int_s^t P_\tau \frac{y_\tau - u_1}{T - \tau} d\tau \right\|^2 \right] &\leq \left(\int_s^t \mathbb{E} \left[\frac{\|P_\tau(y_\tau - u_1)\|^2}{(T - \tau)^2} \right]^{\frac{1}{2}} ds \right)^2 \\ &\leq c(1 + \sqrt{T - s} \mathbb{E}[\|L(u - u_1)\|^2]) \left(\int_s^t \frac{d\tau}{\sqrt{T - \tau}} \right)^2 = 4c(T - s)(1 + \sqrt{T - s} \mathbb{E}[\|L(u - u_1)\|^2]) \end{aligned}$$

Finally using $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$, we have on $(T - \varepsilon_1, T)$

$$\mathbb{E}[\|y_s - y_t\|^2] \leq (C_2 \wedge 4c)(T - s)(2 + \sqrt{T - s} \mathbb{E}[\|L(u - u_1)\|^2])$$

Using Doob's inequality for $s, t \in [0, T - \varepsilon_1]$ with $s < t$

$$\mathbb{E}[\|y_s - y_t\|^2] \leq C_2(t - s)$$

this gives the second result (21). □

Lemma 8. *Let y and z be two bridges, solutions of (3) with $N = 1$ and different initializations. Then, there exist two constants $C > 0$ and $0 < \alpha < 1$ such that for all $t \in [0, T]$*

$$\mathbb{E}[\|y_t - z_t\|^2] \leq C\mathbb{E}[\|y_0 - z_0\|^2]^\alpha \quad (25)$$

Proof. Using their definition, we have on $[T - \varepsilon_1, T]$

$$d(y_t - z_t) = [b_t(y_t) - b_t(z_t)]dt + [\sigma_t(y_t) - \sigma_t(z_t)]d\tilde{w}_t - \frac{P_t(y_t)(y_t - u_1) - P_t(z_t)(z_t - u_1)}{T - t}dt$$

Thus

$$d\|y_t - z_t\|^2 = 2(y_t - z_t)^* \left[[b_t(y_t) - b_t(z_t)]dt + [\sigma_t(y_t) - \sigma_t(z_t)]d\tilde{w}_t - \frac{P_t(y_t)(y_t - u_1) - P_t(z_t)(z_t - u_1)}{T - t}dt \right] + \sum_{i,j} (\sigma_t(y_t) - \sigma_t(z_t))_{i,j}^2 dt \quad (26)$$

In a same way, on $[0, T - \varepsilon_1]$ we obtain

$$d\|y_t - z_t\|^2 = 2(y_t - z_t)^* \left[[b_t(y_t) - b_t(z_t)]dt + [\sigma_t(y_t) - \sigma_t(z_t)]d\tilde{w}_t \right] + \sum_{i,j} (\sigma_t(y_t) - \sigma_t(z_t))_{i,j}^2 dt \quad (27)$$

We denote $E_t = \mathbb{E}[\|y_t - z_t\|^2]$. We decompose the interval $[0, T]$ into $[0, T - h]$ and $(T - h, T]$ with a parameter h that will be chosen later. On $[0, T - h]$ with respect to both precedent Inequations (26) and (27) we get by using regularity of b and σ

$$E'_t \leq C_1(E_t + \frac{E_t}{T - t}) \leq \frac{C_2 E_t}{T - t}$$

where C_1 and C_2 are positive constants depending on T , b and σ . We use Gronwall's lemma to obtain

$$E_t \leq E_0 \left(\frac{T}{h} \right)^{C_2}$$

For the other part $(T - h, T]$, we use (26), (27) and (20) to get

$$E'_t \leq C_3(E_t + \sqrt{\frac{E_t}{T - t}}) \leq C_4 \frac{E_t + 1}{\sqrt{T - t}}$$

where C_3 and C_4 are positive constants depending on T , b and σ . Then

$$\log(E_t + 1) - \log(E_{T-h} + 1) \leq C_4(\sqrt{h} - \sqrt{T - t}) \leq C_4\sqrt{h}$$

Finally, on $[0, T]$

$$E_t \leq E_0 \left(\frac{T}{h} \right)^{C_5} (1 + e^{C_5\sqrt{h}}) + e^{C_5\sqrt{h}} - 1 \leq E_0 \left(\frac{T}{h} \right)^{C_5} (1 + K) + C_5 K \sqrt{h}$$

where K is a positive constant depending on T and $C_5 = C_2 \vee C_4$. We then choose $h = \left(\frac{2E_0 T^{C_5}(K+1)}{K} \right)^{\frac{1}{C_5+1}}$ which minimizes the last member above. Hence

$$E_t \leq C E_0^{\frac{1}{2C_5+1}}$$

where C is a positive constant depending on b , σ , P and T . □

Proof of Lemma 2 . We now consider an interval of type $[T_{k-1}, T_k)$. We introduce a process y^k solution on this interval for the Bridge Equation (3) initialized at time T_{k-1} by the value $y_{T_{k-1}}^\varepsilon$. A picture to visualize what is going on is given by Figure 1 in page 17. We use this new process y^k to write

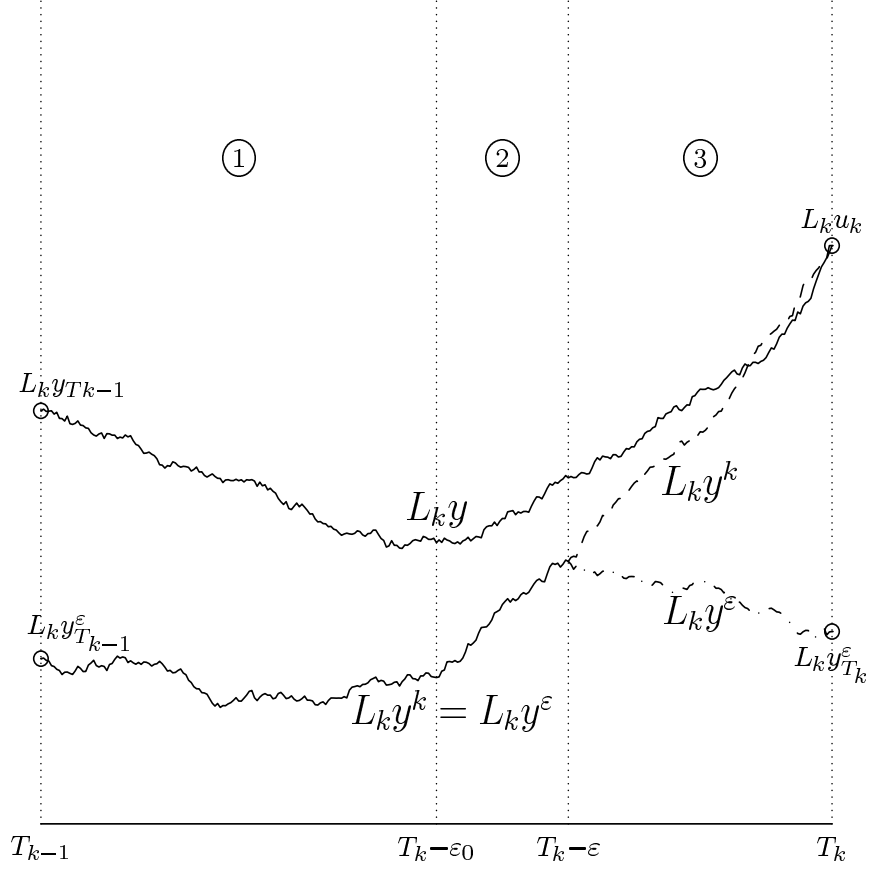
$$\mathbb{E}[\|y_t^\varepsilon - y_t\|^2] \leq 2\mathbb{E}[\|y_t^\varepsilon - y_t^k\|^2] + 2\mathbb{E}[\|y_t^k - y_t\|^2] \quad (28)$$

We will study both terms separately.

For the first one, on $[T_{k-1}, T_k - \varepsilon)$ the term $\|y_t^\varepsilon - y_t^k\|$ is 0 a.s. and on $[T_k - \varepsilon, T_k)$, we can reduce the study to that of $\|x_t - y_t\|^2$ with a same initialization $y_{T_k - \varepsilon}^\varepsilon$ at time $T_k - \varepsilon$.

$$\mathbb{E}[\mathbb{E}[\|y_t^\varepsilon - y_t^k\|^2 | y_{T_k - \varepsilon}^\varepsilon]] \leq 2\mathbb{E}[\mathbb{E}[\|y_{T_k - \varepsilon}^k - y_t^k\|^2 | y_{T_k - \varepsilon}^\varepsilon]] + 2\mathbb{E}[\mathbb{E}[\|y_t^\varepsilon - y_{T_k - \varepsilon}^\varepsilon\|^2 | y_{T_k - \varepsilon}^\varepsilon]]$$

Figure 1: Illustration of the three different dynamics considered



Let us recall

$$dx = b_t(x_t)dt + \sigma_t(x_t)dw_t \quad (1)$$

$$dy = b_t(y_t)dt - P_t(y_t)\frac{y_t - u_k}{T_k - t} + \sigma_t(y_t)dw_t \quad (3)$$

- ① First, the three processes follow the dynamics of the initial diffusion (1) with a different initialization for y .
- ② Now, the three processes follow the dynamics of the bridge (3), that means that the correction term operates and forces these processes to get closer to the observation
- ③ At the end, only the processes y and y^k go on following the dynamics of the bridge (3) and both tends to the observation, while y^ε follows the initial dynamics (1)

We then use Lemma (7) given in the appendix and classical technique (see *e.g.* [5] p.170) to obtain upper bounds

$$\mathbb{E}[\mathbb{E}[\|y_t^\varepsilon - y_t^k\|^2 | y_{T_k-\varepsilon}^\varepsilon] \leq c\varepsilon(1 + \sqrt{\varepsilon} \mathbb{E}[\|L_k(y_{T_k-\varepsilon}^\varepsilon - u_k)\|^2]) \quad (29)$$

where c is a positive constant. Now in order to treat the remaining term we use Lemma 8

$$\mathbb{E}[\mathbb{E}[\|y_t^k - y_t\|^2 | y_{T_k-\varepsilon}^\varepsilon, y_{T_k-\varepsilon}^\varepsilon]] \leq \mathbb{E}[\|y_{T_k-\varepsilon}^\varepsilon - y_{T_k-\varepsilon}\|^2]^\alpha$$

Finally, on $[T_{k-1}, T_k)$

$$\mathbb{E}[\|y_t^\varepsilon - y_t\|^2] \leq c' \left[\varepsilon(1 + \sqrt{\varepsilon} \mathbb{E}[\|L_k(y_{T_k-\varepsilon}^\varepsilon - u_k)\|^2]) + \mathbb{E}[\|y_{T_k-\varepsilon}^\varepsilon - y_{T_k-\varepsilon}\|^2]^\alpha \right] \quad (30)$$

where $0 < \alpha < 1$ and c' is a positive constant only depending on T , bounds b and σ . We show by induction that there exists some constant C such that for all $1 \leq k \leq N$

$$\mathbb{E}[\|y_{T_k}^\varepsilon - y_{T_k}\|^2] \leq C^k \varepsilon^{\alpha^{k-1}} \quad (31)$$

The base case is given by Equation (29). Indeed on $[0, T_1]$ processes y^1 and y are indistinguishable since they have a same initialization at time 0. Suppose now for some k that Inequality (31) holds. We now use Equation (30) to get

$$\mathbb{E}[\|y_{T_{k+1}}^\varepsilon - y_{T_{k+1}}\|^2] \leq c' \left[\varepsilon(1 + \sqrt{\varepsilon} \mathbb{E}[\|L_{k+1}(y_{T_k}^\varepsilon - u_{k+1})\|^2]) + \mathbb{E}[\|y_{T_k}^\varepsilon - y_{T_k}\|^2]^\alpha \right]$$

Let us recall that $L_k y_{T_k} = L_k u_k$ hence

$$\mathbb{E}[\|L_{k+1}(y_{T_k}^\varepsilon - u_{k+1})\|^2] \leq \mathbb{E}[\|L_{k+1}(y_{T_k}^\varepsilon - y_{T_k})\|^2] + \|L_{k+1}(u_k - u_{k+1})\|^2 \leq c_k(1 + \mathbb{E}[\|y_{T_k}^\varepsilon - y_{T_k}\|^2])$$

where c_k is a positive constant depending on L_{k+1} , u_k and u_{k+1} . That gives us thanks to the induction hypothesis

$$\mathbb{E}[\|y_{T_{k+1}}^\varepsilon - y_{T_{k+1}}\|^2] \leq c'_k \varepsilon(1 + \sqrt{\varepsilon} \mathbb{E}[\|y_{T_k}^\varepsilon - y_{T_k}\|^2]) + \mathbb{E}[\|y_{T_k}^\varepsilon - y_{T_k}\|^2]^\alpha \leq C[\varepsilon(1 + \sqrt{\varepsilon} C^k \varepsilon^{\alpha^{k-1}}) + C^k \varepsilon^{\alpha^k}]$$

where C is a positive constant. This concludes the proof. \square

Lemma 9. Let $(t_{k,q})_{\substack{1 \leq k \leq N \\ 1 \leq q \leq M_k}}$ be a sequence such that $t_{k,q} \in (T_{k-1}, T_k)$ and for all k , $(t_{k,q})_q$ is an increasing sequence. Then for all bounded continuous function g

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbb{E}[g(x_{t_{1,1}}, \dots, x_{t_{N,M_N}}) \psi^\varepsilon]}{\mathbb{E}[\psi^\varepsilon]} = \mathbb{E}[g(x_{t_{1,1}}, \dots, x_{t_{N,M_N}}) | (L_k x_{T_k} = v_k)_{1 \leq k \leq N}]$$

Proof. Let us recall

$$C^\varepsilon \psi^\varepsilon = \prod_{k=1}^N \varepsilon^{-\frac{m_k}{2}} \eta_k^\varepsilon(x_{T_k-\varepsilon}) \exp\left\{-\frac{\|\beta_{T_k-\varepsilon}^k(x_{T_k-\varepsilon})(L_k x_{T_k-\varepsilon} - v_k)\|^2}{2\varepsilon}\right\}$$

where for all $z \in \mathbb{R}^n$

$$\eta_k^\varepsilon(z) = \sqrt{\det(A_{T_k-\varepsilon}^k(z))}$$

Let introduce Aronson's estimates (see *e.g.* [1], [8] or [2]) that gives bounds for the transition density. If $p_{s,t}(u, \cdot)$ (with $s < t$) is the density of x_t knowing that $x_s = u$, we have for all z

$$\mu(t-s)^{-\frac{n}{2}} e^{-\frac{\lambda \|z-u\|^2}{t-s}} < p_{s,t}(u, z) < M(t-s)^{-\frac{n}{2}} e^{-\frac{\Lambda \|z-u\|^2}{t-s}}$$

The transition densities allow to expand the density q^ε of $(x_{t_{1,1}}, \dots, x_{t_{N,M_N}}, x_{T_1-\varepsilon}, \dots, x_{T_N-\varepsilon})$

$$q^\varepsilon(z_{1,1}, \dots, z_{t_{N,M_N}}, \zeta_1, \dots, \zeta_N) = p_{0,t_{1,1}}(u, z_{1,1}) \dots p_{t_{1,M_1}, T_1-\varepsilon}(z_{0,M_0}, \zeta_1) \dots p_{t_{N,M_N}, T_N-\varepsilon}(z_{N,M_N}, \zeta_N)$$

Then we set for $\varepsilon \geq 0$

$$\begin{aligned}\Phi_g^\varepsilon(\zeta_1, \dots, \zeta_N) &= \mathbb{E}[g(x_{t_{1,1}}, \dots, x_{t_{N,M_N}}) | (x_{T_k-\varepsilon} = \zeta_k)_k] \\ &= \int g(z_{1,1}, \dots, z_{t_{N,M_N}}) q^\varepsilon(z_{1,1}, \dots, z_{t_{N,M_N}}, \zeta_1, \dots, \zeta_N) \prod_j dz_j\end{aligned}$$

This application is continuous according to Aronson's estimates. From this expression it comes

$$\frac{I_g^\varepsilon}{I_1^\varepsilon} := \frac{\mathbb{E}[g(x_{t_{1,1}}, \dots, x_{t_{N,M_N}}) C^\varepsilon \psi^\varepsilon]}{\mathbb{E}[C^\varepsilon \psi^\varepsilon]} = \frac{C^\varepsilon \int \Phi_g^\varepsilon \prod_k \eta_k^\varepsilon \exp\{-\frac{\|\beta_{T_k-\varepsilon}^k(L_k \zeta_k - v_k)\|^2}{2\varepsilon}\} d\zeta_k}{C^\varepsilon \int \Phi_1^\varepsilon \prod_k \eta_k^\varepsilon \exp\{-\frac{\|\beta_{T_k-\varepsilon}^k(L_k \zeta_k - v_k)\|^2}{2\varepsilon}\} d\zeta_k}$$

We recall that the rows of each matrix L_k form an orthonormal family. We now complete arbitrarily each family into an orthonormal basis of \mathbb{R}^n . We denote P_k an arbitrary matrix whose first rows are given by L_k . Then we make a basis change with respect to those matrices P_k for each ζ_k . Thus

$$I_g^\varepsilon = C^\varepsilon \int \Phi_g^\varepsilon(P_1^{-1}\zeta_1, \dots, P_N^{-1}\zeta_N) \prod_k \eta_k^\varepsilon(P_k^{-1}\zeta_k) \exp\{-\frac{\|\beta_{T_k-\varepsilon}^k(\zeta_k^{1:m_k} - v_k)\|^2}{2\varepsilon}\} d\zeta_k$$

denoting $\zeta_k^{i:j}$ the vector composed by the coordinates from i^{th} to j^{th} one of ζ_k . We now make a second change

$$\begin{cases} \zeta_k^{1:m_k} = \sqrt{\varepsilon} \xi_k^{1:m_k} + v_k \\ \zeta_k^{m_k+1:n} = \xi_k^{m_k+1:n} \end{cases}$$

So that

$$\begin{aligned}I_g^\varepsilon &= \left(\prod_k \varepsilon^{-\frac{m_k}{2}} \right) \int \Phi_g^\varepsilon(P_1^{-1}\zeta_1, \dots, P_N^{-1}\zeta_N) \prod_k \eta_k^\varepsilon \exp\{-\frac{\|\beta_{T_k-\varepsilon}^k(\zeta_k^{1:m_k} - v_k)\|^2}{2\varepsilon}\} d\zeta_k \\ &= \int \Phi_g^\varepsilon \left(P_1^{-1} \begin{pmatrix} \sqrt{\varepsilon} \xi_1^{1:m_k} + v_1 \\ \xi_1^{m_k+1:n} \end{pmatrix}, \dots, (P_1^{-1} \begin{pmatrix} \sqrt{\varepsilon} \xi_N^{1:m_k} + v_N \\ \xi_N^{m_k+1:n} \end{pmatrix}) \right) \prod_k \eta_k^\varepsilon \exp\{-\frac{\|\beta_{T_k-\varepsilon}^k \xi_k^{1:m_k}\|^2}{2}\} d\xi_k\end{aligned}$$

We now use Aronson's estimates and Lemma 11 to get an integrable uniform upper bound for q^ε when $0 < \varepsilon < \varepsilon_0$. Thanks to Lebesgue's theorem we obtain the convergence for the last term

$$I_g^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \int \Phi_g^0 \left(P_1^{-1} \begin{pmatrix} v_1 \\ \xi_1^{m_k+1:n} \end{pmatrix}, \dots, (P_1^{-1} \begin{pmatrix} v_N \\ \xi_N^{m_k+1:n} \end{pmatrix}) \right) \prod_k \eta_k^\varepsilon \exp\{-\frac{\|\beta_{T_k}^k \xi_k^{1:m_k}\|^2}{2}\} d\xi_k$$

We then integrate with respect to the $\xi_k^{1:m_k}$

$$I_g^\varepsilon = \int \Phi_g^0 \left(P_1^{-1} \begin{pmatrix} v_1 \\ \xi_1^{m_k+1:n} \end{pmatrix}, \dots, (P_1^{-1} \begin{pmatrix} v_N \\ \xi_N^{m_k+1:n} \end{pmatrix}) \right) \prod_k d\xi_k^{m_k+1:n}$$

Finally

$$\lim_{\varepsilon \rightarrow 0} \frac{I_g^0}{I_1^0} = \frac{\int \Phi_g^0 \left(P_1^{-1} \begin{pmatrix} v_1 \\ \xi_1^{m_k+1:n} \end{pmatrix}, \dots, (P_1^{-1} \begin{pmatrix} v_N \\ \xi_N^{m_k+1:n} \end{pmatrix}) \right) \prod_k d\xi_k^{m_k+1:n}}{\int \Phi_1^0 \left(P_1^{-1} \begin{pmatrix} v_1 \\ \xi_1^{m_k+1:n} \end{pmatrix}, \dots, (P_1^{-1} \begin{pmatrix} v_N \\ \xi_N^{m_k+1:n} \end{pmatrix}) \right) \prod_k d\xi_k^{m_k+1:n}} \quad (32)$$

We conclude thanks to the Bayes formula. \square

Lemma 10. *Let y be solution of Equation (8). Then almost surely for all $1 \leq k \leq N$ the following integral are absolutely convergent*

$$\begin{aligned}& \int_{T_k-\varepsilon_k}^{T_k} \frac{(L_k y_t - v_k)^* A_t^k(y_t) b_t(y_t) dt}{T_k - t} + \frac{(L_k y_t - v_k)^* dA_t^k(y_t) (L_k y_t - v_k)}{2(T_k - t)} \\ & + \sum_{1 \leq i, j \leq m_k} \frac{d\langle A_{i,j}^k(y), (L_k y_t - v_k)_i (L_k y_t - v_k)_j \rangle_t}{2(T_k - t)} \quad (33)\end{aligned}$$

Proof. We reduce the study without loss of generality to that of

$$dy_t = b_t(y_t)dt + \sigma_t(y_t)d\tilde{w}_t - \sigma_t(y_t)\beta_t(y_t)\frac{Ly_t - v}{T - t}\mathbf{1}_{(T-\varepsilon_1, T)}(t)dt$$

We then treat integrability for each term.

For the first term, since b and β are bounded, we use Lemma 1 to get

$$\left\| \frac{Ly_t - v}{T - t} \right\| \leq C \sqrt{\frac{\log \log ((T - t)^{-1} + e)}{T - t}}$$

where C is a positive random variable. Now for all positive α , we have $\log \log x \leq x^\alpha$. Then for α small enough, we obtain integrability of righthandside.

For the second term in (33), we recall that for all z we have $A_t(z) = \beta_t(z)^* \beta_t(z) = (La_t(z)L^*)^{-1}$, hence

$$dA_t = p_t dt + q_t d\tilde{w}_t + r_t \frac{Ly_t - v}{T - t} dt$$

where p , q and r are bounded adapted processes. So that even if it means changing p , q and r

$$\frac{(Ly_t - v)^* dA_t (Ly_t - v)}{T - t} = \frac{\|Ly_t - v\|^2}{T - t} p_t dt + \frac{\|Ly_t - v\|^2}{T - t} q_t d\tilde{w}_t + \frac{\|Ly_t - v\|^2}{(T - t)^2} r_t dt \quad (34)$$

Using Lemma 1, we obtain that the quantities $\frac{\|Ly_t - v\|^2}{T - t}$, $\frac{\|Ly_t - v\|^2}{(T - t)^2}$ and $\frac{\|Ly_t - v\|^4}{(T - t)^2}$ are integrable in a left neighborhood of T .

For the last term in (33), we use Itô's formula and the fact that $L\sigma_t(z)\beta_t(z) = I_d$, so that on $(T - \varepsilon_k, T_k)$

$$d(Ly_t - v) = L[b_t dt + \sigma_t d\tilde{w}_t] - \frac{Ly_t - v}{T - t} dt$$

Hence

$$d\langle A_{i,j}, (Ly_t - v)_i (Ly_t - v)_j \rangle_t \leq \|Ly_t - v\| p_t dt$$

where p is the same bounded adapted process given above. Finally

$$\sum_{i,j} \frac{d\langle A_{i,j}, (Ly_t - v)_i (Ly_t - v)_j \rangle_t}{T - t} \leq \frac{\|Ly_t - v\| p_t}{T - t} dt$$

even if it means changing p , and this last term is integrable. \square

Lemma 11. Let $(Z_j)_{1 \leq j \leq K}$ be a family of random m_j -dimensional variables and let $(g_j : \mathbb{R}^{m_j} \rightarrow \mathbb{R})_{1 \leq j \leq K}$ be a family of densities. Then the function

$$\begin{aligned} \prod_{j=1}^K \mathbb{R}^{m_j} &\rightarrow \mathbb{R} \\ (v_j)_j &\mapsto \mathbb{E} \left[\prod_j g_j(Z_j - v_j) \right] \end{aligned}$$

is the density of the family $(V_j = W_j + Z_j)_{1 \leq j \leq K}$ where each of the W_j whose law is given by g_j is independent with respect to the $(Z_j)_{1 \leq j \leq K}$ and $(W_k)_{k \neq j}$.

Proof. Let f be a bounded continuous function

$$\int f((v_j)_j) \mathbb{E} \left[\prod_j g_j(Z_j - v_j) \right] \prod_j dv_j = \mathbb{E} \left[\int f((v_j)_j) \prod_j g_j(Z_j - v_j) \right] \prod_j dv_j$$

Then we make the change of variables $w_j = Z_j - v_j$ for all j

$$\int f((v_j)_j) \mathbb{E} \left[\prod_j g_j(Z_j - v_j) \right] \prod_j dv_j = \mathbb{E} \left[\int f((w_j + Z_j)_j) \prod_j g_j(w_j) \right] \prod_j dw_j$$

Hence

$$\mathbb{E} \left[\int f((w_j + Z_j)_j) \prod_j g_j(w_j) \right] \prod_j dw_j = \mathbb{E} [f((W_j + Z_j)_j)]$$

where W_j admits g_j as density. \square

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